MATHEMATICS

CALCULATION OF THE VOLUME OF INTERSECTION OF A SPHERE AND A CYLINDER

BY

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1. *Introduction*

From an investigation into spherical interstellar-hydrogen clouds the following mathematical problem arose: Given a sphere, which is intersected by a right circular cylinder, determine the volume of the intersection. In the present article we express this volume in terms of elliptic integrals and report on our numerical results.

The volume of the intersection depends on three variables, viz. the radii of the sphere and the cylinder and the distance of the centre of the sphere to the axis of the cylinder. We take the radius of the sphere equal to unity and use for the two remaining variables the following symbols:

ϱ, radius of the cylinder;

η, distance of the centre of the sphere to the axis of the cylinder.

Further, $V(\varrho, \eta)$ will denote the volume of the intersection.

For the calculation of $V(\varrho, \eta)$ we must distinguish different cases, which are characterized by the mutual position of the sphere and the cylinder. In fig. 1 we have drawn the section of the sphere and the cylinder with the plane through the centre of the sphere orthogonal to the axis of the cylinder; M_s denotes the centre of the sphere, M_c the intersection of the plane with the axis of the cylinder.

It is easy to see that the cases $\eta-\varrho \leqq -1$ and $\eta-\varrho \geqq 1$ may be ignored: if $\eta-\varrho \leqq -1$ the sphere lies completely inside the cylinder, hence $V(\varrho, \eta)=\frac{4}{3}\pi$; and if $\eta-\varrho \geqq 1$ the sphere lies completely outside the cylinder, hence $V(\varrho, \eta)=0$. Therefore we henceforth suppose:

$$|\eta-\varrho|<1. \tag{1}$$

The remaining cases may be divided in two ways:

First, I. $\eta+\varrho>1$. The circle (M_c, ϱ) intersects the circle $(M_s, 1)$ at two points.

II. $\eta+\varrho<1$. The circle (M_c, ϱ) lies completely inside the circle $(M_s, 1$

III. $\eta+\varrho=1$. The circle (M_c, ϱ) touches the circle $(M_s, 1)$ innerly.

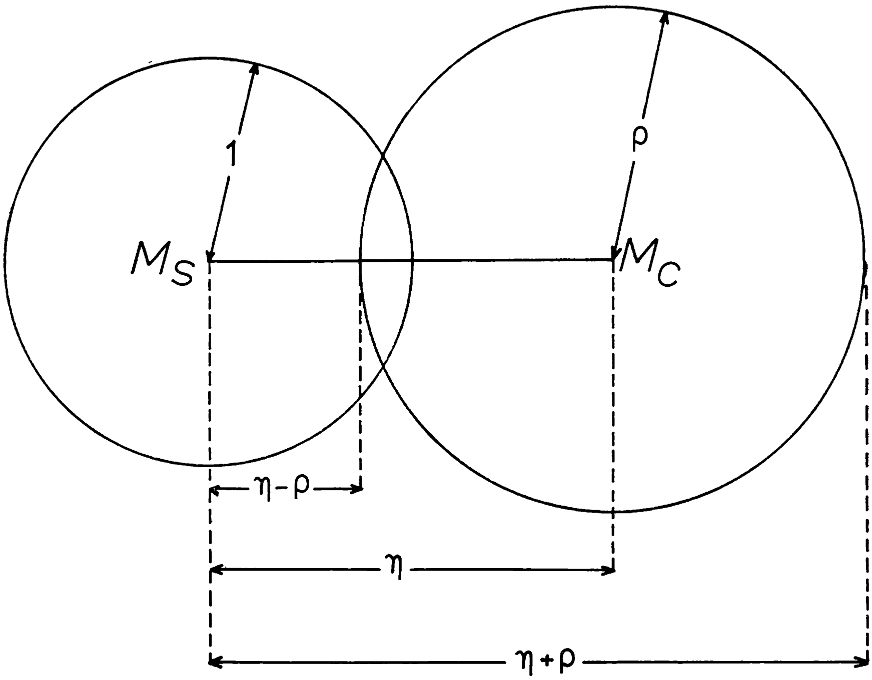

Fig. 1. Section of the sphere and the cylinder with the plane through the centre of the sphere perpendicular to the axis of the cylinder.

Secondly, A. $\eta > \varrho$. M_s lies outside the circle (M_c, ϱ).

B. $\eta < \varrho$. M_s lies inside the circle (M_c, ϱ).

C. $\eta = \varrho$. The circle (M_c, ϱ) passes through M_s.

In fig. 2 a diagram of the different cases has been drawn.

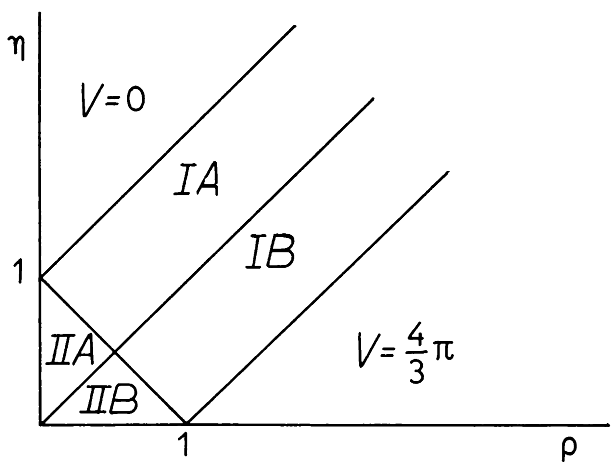

Fig. 2. Cases IA and IB: V is given by formula (5). Cases IIA and IIB: V is given by formula (8).

2. *Case I;* $\eta+\varrho>1$

A. $\eta>\varrho$. In this case the sphere and the cylinder are in the position as shown in fig. 1. For the calculation of $V(\varrho, \eta)$ we introduce polar coordinates r and φ with centre M_s. Then we can write

$$V(\varrho, \eta)=4 \int_{\eta-\varrho}^{1} \sqrt{1-r^2}\; r\, dr \int_{0}^{\varphi(r)} d\varphi, \tag{2}$$

in which

$$\varphi(r)=\cos^{-1} \frac{\eta^2+r^2-\varrho^2}{2\eta r}. \tag{3}$$

After performing the integration with respect to φ, we derive by means of partial integration with respect to r:

$$V(\varrho, \eta)=\left\{-\frac{4}{3}(1-r^2)^{3/2} \cos^{-1} \frac{\eta^2+r^2-\varrho^2}{2\eta r}\right\}_{\eta-\varrho}^{1} -$$

$$-\frac{4}{3} \int_{\eta-\varrho}^{1} \frac{(1-r^2)^{3/2}(r^2-\eta^2+\varrho^2)}{r\sqrt{\{r^2-(\eta-\varrho)^2\}\{(\eta+\varrho)^2-r^2\}}}\, dr =$$

$$=-\frac{2}{3} \int_{(\eta-\varrho)^2}^{1} \frac{(1-t)^{3/2}(t-\eta^2+\varrho^2)}{t\sqrt{\{t-(\eta-\varrho)^2\}\{(\eta+\varrho)^2-t\}}}\, dt =$$

$$=-\frac{2}{3} \int_{(\eta-\varrho)^2}^{1} \frac{t^2+t(\varrho^2-\eta^2-2)+(2\eta^2-2\varrho^2+1)+(\varrho^2-\eta^2)/t}{\sqrt{\{(\eta+\varrho)^2-t\}(1-t)\{t-(\eta-\varrho)^2\}}}\, dt.$$

By means of [1] (formulae (234.00), (234.16) and (234.17)) the following expression for $V(\varrho, \eta)$ can be derived:

$$\left\{\begin{aligned} V(\varrho, \eta)= & -\frac{2}{3\sqrt{\varrho\eta}}\left[\frac{8\varrho^2\eta}{3(\eta+\varrho)}(3\varrho^2+4\varrho\eta+\eta^2-3)(1-k^2)K(k)-\right. \\ & \left. -\frac{4}{3}\varrho\eta(7\varrho^2+\eta^2-4)E(k)-\frac{4\varrho\eta(\eta-\varrho)(1-k^2)}{\eta+\varrho}\Pi((\eta+\varrho)^2k^2, k)\right], \end{aligned}\right. \tag{4}$$

in which

$$k^2=\frac{1-(\eta-\varrho)^2}{4\varrho\eta};$$

$K(k)$, $E(k)$ and $\Pi((\eta+\varrho)^2k^2, k)$ denote Legendre's complete elliptic integrals of the first, second and third kinds, respectively (cf. [1], form. (110.06), (110.07) and (110.08)). The complete elliptic integral of the third kind $\Pi((\eta+\varrho)^2k^2, k)$ may be expressed in terms of complete and incomplete elliptic integrals of the first and second kinds by means of HEUMAN's lambda function, [2]. According to [1] (form. (413.01)) we have

$$\frac{4\varrho\eta(\eta-\varrho)(1-k^2)}{\eta+\varrho}\Pi((\eta+\varrho)^2k^2, k) = \pi\sqrt{\varrho\eta}\,\Lambda_0(\xi, k),$$

in which

$$\xi=\sin^{-1}\frac{2\sqrt{\varrho\eta}}{\eta+\varrho}$$

and

$$\Lambda_0(\xi, k) = \frac{2}{\pi} [E(k)F(\xi, k') + K(k)E(\xi, k') - K(k) F(\xi, k')]$$

(cf. [1], form. (150.03)).

To this Heuman function we apply the special addition formula [1], form. (153.01). Finally, this expression is substituted in (4), which yields the following formula for $V(\varrho, \eta)$:

$$(5)\quad \left\{ \begin{aligned} V(\varrho, \eta) = \frac{2}{3}\pi\{1-\Lambda_0(\vartheta, k)\} - \frac{8}{9}\sqrt{\varrho\eta}(6\varrho^2+2\varrho\eta-3)(1-k^2)K(k) + \\ + \frac{8}{9}\sqrt{\varrho\eta}(7\varrho^2+\eta^2-4)E(k), \end{aligned} \right.$$

in which

$$\vartheta = \sin^{-1}(\eta-\varrho) \text{ and } k^2 = \frac{1-(\eta-\varrho)^2}{4\varrho\eta}.$$

B. $\eta < \varrho$. In this case the volume of the intersection is given by

$$V(\varrho, \eta) = 4\int_0^{\varrho-\eta} \sqrt{1-r^2}\, r\, dr \int_0^{\pi} d\varphi + 4 \int_{\varrho-\eta}^{1} \sqrt{1-r^2}\, r\, dr \int_0^{\varphi(r)} d\varphi$$

with $\varphi(r)$ given by (3).

After a similar derivation as in case I A we arrive at the same formula (5) for $V(\varrho, \eta)$. Note that in this case ϑ is negative, but owing to [1] (form. (151.01))

$$\Lambda_0(\vartheta, k) = -\Lambda_0(-\vartheta, k).$$

C. $\eta = \varrho$. It is evident that this case must follow continuously from the cases I A and I B. Hence, $V(\varrho, \eta)$ is again represented by formula (5), which now can be simplified to the following form:

$$(6)\qquad V(\eta, \eta) = \frac{2}{3}\pi - \frac{2(4\eta^2-1)(8\eta^2-3)}{9\eta} K\left(\frac{1}{2\eta}\right) + \frac{32}{9}\eta(2\eta^2-1)E\left(\frac{1}{2\eta}\right),$$

a result which may also be derived by direct calculation.

As a check to formula (5) we consider the following three special cases:

1. Substitution of $\eta-\varrho = -1$ in (5) yields $V(\varrho, \eta) = \frac{4}{3}\pi$.
2. Substitution of $\eta-\varrho = 1$ in (5) yields $V(\varrho, \eta) = 0$.
3. Taking the limit $\eta \to \infty$, $\varrho \to \infty$ with $\eta-\varrho = 1-h$, h constant, in formula (5) yields

$$(7)\qquad \lim_{\substack{\varrho\to\infty,\, \eta\to\infty \\ \eta-\varrho=1-h}} V(\varrho, \eta) = \pi(h^2 - \tfrac{1}{3}h^3),$$

which agrees with the expression for the volume of a segment of a unit sphere with height h.

3. *Case II*; $\eta+\varrho<1$

A. $\eta>\varrho$. By means of a figure analogous to fig. 1 we easily derive

$$V(\varrho, \eta)=4 \int_{\eta-\varrho}^{\eta+\varrho} \sqrt{1-r^2}\, r\, dr \int_0^{\varphi(r)} d\varphi,$$

with $\varphi(r)$ given by (3).

In the same way as in case I A, we find

$$(8)\quad \left\{\begin{aligned} V(\varrho, \eta)&=\frac{2}{3}\pi\{1-\Lambda_0(\vartheta, k)\}- \\ &-\frac{4\sqrt{1-(\eta-\varrho)^2}}{9(\eta+\varrho)}\{2\varrho-4\eta+(\eta+\varrho)(\eta-\varrho)^2\}(1-k^2)K(k)+ \\ &+\frac{4}{9}\sqrt{1-(\eta-\varrho)^2}\,(7\varrho^2+\eta^2-4)E(k),\end{aligned}\right.$$

in which

$$\vartheta=\sin^{-1}\frac{\eta-\varrho}{\eta+\varrho} \text{ and } k^2=\frac{4\varrho\eta}{1-(\eta-\varrho)^2}.$$

B. $\eta<\varrho$. In this case the volume of intersection is given by

$$V(\varrho, \eta)=4 \int_0^{\varrho-\eta} \sqrt{1-r^2}\, r\, dr \int_0^{\pi} d\varphi+4 \int_{\varrho-\eta}^{\varrho+\eta} \sqrt{1-r^2}\, r\, dr \int_0^{\varphi(r)} d\varphi,$$

with $\varphi(r)$ given by (3).

After some reductions we arrive at the same formula (8) for $V(\varrho, \eta)$.

C. $\eta=\varrho$. In this case the same formula (8) holds for the volume $V(\varrho, \eta)$. This formula may be simplified to

$$(9)\quad V(\eta, \eta)=\frac{2}{3}\pi+\frac{4}{9}(1-4\eta^2)K(2\eta)+\frac{16}{9}(2\eta^2-1)E(2\eta),$$

a result which may also be derived by direct calculation.

As a check to formula (8) we consider the following special cases:

1. $\eta=0$, $0\leqq\varrho\leqq1$. In this case the axis of the cylinder passes through the centre of the sphere. Substitution of η in formula (8) yields

$$(10)\quad V(\varrho, 0)=\frac{4}{3}\pi-\frac{4}{3}\pi(1-\varrho^2)^{3/2}$$

which can easily be verified by direct calculation.

2. $\varrho=0$, $0\leqq\eta\leqq1$. In this case the cylinder reduces to a line. Substitution of ϱ in formula (8) yields the correct result, $V(0, \eta)=0$.

4. *Case III*; $\eta+\varrho=1$

The expression for $V(\varrho, \eta)$ in this case must follow continuously from the formulae (5) and (8). Taking the limit for $\eta+\varrho \to 1$ in these formulae yields

$$V(1-\eta, \eta)=\frac{4}{3}\left\{\frac{\pi}{2}-\sin^{-1}(2\eta-1)\right\}+\frac{8}{9}\sqrt{\eta(1-\eta)}(8\eta^2-14\eta+3), \tag{11}$$

which can be verified by direct calculation.

As a check to formula (11) we substitute $\eta=\frac{1}{2}$ which yields

$$V(\tfrac{1}{2}, \tfrac{1}{2})=\frac{2}{3}\pi-\frac{8}{9}, \tag{12}$$

which result agrees with the result derived from taking the limit $\eta \to \frac{1}{2}$ in formulae (6) and (9) or from direct calculation.

5. *Integral representation of the volume of the intersection*

In this section we show that the volume of the intersection of the sphere and the cylinder can be represented as an infinite integral of a product of three Bessel functions. We therefore introduce a function $f(x)$ defined by

$$f(x)=\varrho \int_0^\infty J_0(xt)\, J_1(\varrho t)\, dt. \tag{13}$$

Owing to [3] (form. 13.42(9)) we have

$$f(x)=\begin{cases} 0 & \text{for } |x|>|\varrho|, \\ \frac{1}{2} & \text{for } |x|=|\varrho|, \\ 1 & \text{for } |x|<|\varrho|. \end{cases} \tag{14}$$

Hence, when we interpret x as a coordinate denoting the distance to the axis of the cylinder, the function $f(x)$ will be equal to zero outside the cylinder and equal to one inside the cylinder. The volume of the intersection may now be derived by integrating the function $f(x)$ over the volume of the sphere. Using the same polar coordinates r and φ as in section 2 the following expression for $V(\varrho, \eta)$ is obtained

$$\begin{cases} V(\varrho, \eta)=2 \int_0^1 r\sqrt{1-r^2}\, dr \int_0^{2\pi} f(\sqrt{r^2+\eta^2-2r\eta \cos \varphi})\, d\varphi= \\ \quad =2\varrho \int_0^1 r\sqrt{1-r^2}\, dr \int_0^{2\pi} d\varphi \int_0^\infty J_0(t\sqrt{r^2+\eta^2-2r\eta \cos \varphi})J_1(\varrho t)\, dt. \end{cases} \tag{15}$$

When we change the order of the last two integrations, the integration with respect to φ can be performed, [3] (form. 11.41(16)), so that

$$V(\varrho, \eta)=4\pi\varrho \int_0^1 r\sqrt{1-r^2}\, dr \int_0^\infty J_0(rt) J_0(\eta t) J_1(\varrho t)\, dt. \tag{16}$$

When we change again the order of the integrations and perform the integration with respect to r ([3], form. 12.11(1)) we arrive at the following integral representation for $V(\varrho, \eta)$:

$$V(\varrho, \eta) = 2\pi\varrho\sqrt{2\pi}\int_0^\infty J_1(\varrho t)\, J_0(\eta t)\, \frac{J_{3/2}(t)}{t^{3/2}}\, dt. \tag{17}$$

The connection of this representation with our previous results is shown in the following way. First, for some special values of the variables ϱ and η we make use of two formulae derived by BAILEY, [4] (form. (8.2) and (8.3)) which yield

$$V(\varrho, \eta) = \frac{4}{3}\pi \quad \text{for } \eta > \varrho + 1 \text{ or } \varrho - \eta < -1,$$
$$V(\varrho, \eta) = 0 \quad \text{for } \varrho > \eta + 1 \text{ or } \varrho - \eta > 1,$$

in accordance with the remark made in section 1.

Secondly it may be shown that in the case $\eta = \varrho$ the integral (17) may be partially integrated yielding a simpler integral, which can be expressed in terms of elliptic integrals by means of [3] (form. 13.46(1)) and of appropriate formulae of [1]. The result agrees with the formulae (6) and (9).

However, in the general case of arbitrary values of ϱ and η, only subject to (1), the integral (17) does not seem very suitable for further computation. We may better start from (16), in which we perform the integration to t ([3], form. 13.46(2)). We then obtain the following integral for $V(\varrho, \eta)$:

$$V(\varrho, \eta) = 4\int_0^1 r\sqrt{1-r^2}\, A\, dr, \tag{18}$$

$$\text{in which } A = \begin{cases} 0 & \text{for } \varrho < |\eta - r|, \\ \cos^{-1}\dfrac{\eta^2 + r^2 - \varrho^2}{2\eta r} & \text{for } |\eta - r| < \varrho < \eta + r, \\ \pi & \text{for } \varrho > \eta + r. \end{cases}$$

It is easy to see that this integral is the same integral with which we started our derivation for the cases I A, I B, II A, II B in sections 2 and 3.

6. *Numerical results*

By means of formulae (5) and (8) we have computed a table of $V(\varrho, \eta)$ to four decimal places, for values of the arguments

$$\varrho = 0(0.05)0.2;\quad \eta = 0(0.025)1 + \varrho$$
$$\varrho = 0.25(0.05)1;\quad \eta = 0(0.025)1.2(0.1)1 + \varrho$$
$$\varrho = 1.05(0.05)1.5;\quad \eta = \varrho - 1(0.025)1.2(0.1)2$$
$$\varrho = 1.55(0.05)3;\quad \eta = \varrho - 1(0.1)2.$$

The computation was performed on our digital computer ZEBRA.

The computed table of $V(\varrho, \eta)$ is given on the following pages.

Table of $V(\rho,\eta)$

η \ ρ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.000	0.0157	0.0627	0.1406	0.2488	0.3865	0.5526	0.7456	0.9640	1.2056	1.4681
0.025	0.0157	0.0627	0.1405	0.2487	0.3864	0.5524	0.7454	0.9636	1.2051	1.4675
0.050	0.0157	0.0626	0.1404	0.2485	0.3860	0.5518	0.7446	0.9626	1.2038	1.4658
0.075	0.0157	0.0625	0.1402	0.2481	0.3854	0.5509	0.7433	0.9609	1.2016	1.4630
0.100	0.0156	0.0624	0.1399	0.2475	0.3845	0.5496	0.7415	0.9584	1.1984	1.4590
0.125	0.0156	0.0622	0.1395	0.2468	0.3833	0.5479	0.7392	0.9553	1.1944	1.4538
0.150	0.0155	0.0620	0.1390	0.2459	0.3819	0.5459	0.7363	0.9515	1.1894	1.4475
0.175	0.0155	0.0617	0.1384	0.2448	0.3802	0.5434	0.7329	0.9470	1.1835	1.4400
0.200	0.0154	0.0614	0.1377	0.2436	0.3783	0.5406	0.7290	0.9417	1.1766	1.4312
0.225	0.0153	0.0611	0.1369	0.2422	0.3761	0.5373	0.7245	0.9357	1.1688	1.4212
0.250	0.0152	0.0607	0.1360	0.2406	0.3736	0.5337	0.7194	0.9289	1.1599	1.4098
0.275	0.0151	0.0602	0.1351	0.2389	0.3708	0.5296	0.7138	0.9213	1.1501	1.3971
0.300	0.0150	0.0598	0.1340	0.2370	0.3678	0.5251	0.7075	0.9130	1.1391	1.3831
0.325	0.0148	0.0592	0.1328	0.2348	0.3644	0.5202	0.7007	0.9037	1.1270	1.3675
0.350	0.0147	0.0587	0.1315	0.2325	0.3607	0.5148	0.6932	0.8937	1.1138	1.3504
0.375	0.0145	0.0581	0.1301	0.2300	0.3568	0.5090	0.6850	0.8826	1.0993	1.3316
0.400	0.0144	0.0574	0.1286	0.2273	0.3524	0.5026	0.6761	0.8707	1.0835	1.3110
0.425	0.0142	0.0567	0.1270	0.2244	0.3478	0.4958	0.6665	0.8577	1.0663	1.2885
0.450	0.0140	0.0559	0.1252	0.2212	0.3428	0.4884	0.6562	0.8436	1.0476	1.2637
0.475	0.0138	0.0551	0.1234	0.2178	0.3374	0.4804	0.6450	0.8284	1.0272	1.2363
0.500	0.0136	0.0542	0.1213	0.2142	0.3316	0.4719	0.6329	0.8118	1.0048	1.2055
0.525	0.0134	0.0533	0.1192	0.2103	0.3254	0.4626	0.6198	0.7939	0.9802	1.1698
0.550	0.0131	0.0522	0.1169	0.2062	0.3187	0.4527	0.6057	0.7743	0.9526	1.1310
0.575	0.0128	0.0512	0.1144	0.2017	0.3116	0.4420	0.5904	0.7527	0.9206	1.0905
0.600	0.0126	0.0500	0.1118	0.1969	0.3039	0.4305	0.5738	0.7285	0.8860	1.0486
0.625	0.0122	0.0488	0.1090	0.1918	0.2956	0.4180	0.5555	0.7005	0.8498	1.0058
0.650	0.0119	0.0475	0.1060	0.1863	0.2867	0.4044	0.5349	0.6703	0.8126	0.9625
0.675	0.0116	0.0461	0.1027	0.1804	0.2770	0.3894	0.5110	0.6387	0.7746	0.9188
0.700	0.0112	0.0445	0.0993	0.1740	0.2664	0.3726	0.4852	0.6063	0.7363	0.8751
0.725	0.0108	0.0429	0.0955	0.1670	0.2548	0.3528	0.4584	0.5733	0.6978	0.8313
0.750	0.0104	0.0412	0.0914	0.1594	0.2416	0.3315	0.4309	0.5402	0.6593	0.7878
0.775	0.0099	0.0393	0.0870	0.1510	0.2260	0.3094	0.4031	0.5070	0.6210	0.7446
0.800	0.0094	0.0372	0.0821	0.1413	0.2091	0.2869	0.3752	0.4740	0.5830	0.7018
0.825	0.0088	0.0349	0.0766	0.1297	0.1917	0.2643	0.3475	0.4413	0.5455	0.6597
0.850	0.0082	0.0324	0.0703	0.1172	0.1741	0.2417	0.3201	0.4092	0.5086	0.6181
0.875	0.0076	0.0295	0.0624	0.1043	0.1566	0.2195	0.2932	0.3776	0.4724	0.5773
0.900	0.0068	0.0260	0.0540	0.0915	0.1394	0.1978	0.2669	0.3467	0.4369	0.5374
0.925	0.0059	0.0215	0.0455	0.0790	0.1226	0.1767	0.2413	0.3166	0.4024	0.4984
0.950	0.0047	0.0168	0.0372	0.0669	0.1064	0.1563	0.2166	0.2874	0.3688	0.4603
0.975	0.0030	0.0123	0.0294	0.0554	0.0910	0.1367	0.1928	0.2593	0.3362	0.4234
1.000	0.0015	0.0082	0.0222	0.0447	0.0765	0.1182	0.1700	0.2322	0.3047	0.3876
1.025	0.0004	0.0048	0.0159	0.0350	0.0630	0.1007	0.1483	0.2062	0.2744	0.3529
1.050	0.0000	0.0022	0.0104	0.0262	0.0506	0.0843	0.1278	0.1815	0.2454	0.3196
1.075		0.0006	0.0060	0.0185	0.0393	0.0691	0.1086	0.1581	0.2177	0.2875
1.100		0.0000	0.0027	0.0121	0.0293	0.0553	0.0908	0.1360	0.1914	0.2569
1.125			0.0007	0.0069	0.0206	0.0429	0.0743	0.1154	0.1665	0.2277
1.150			0.0000	0.0031	0.0134	0.0319	0.0593	0.0963	0.1431	0.2000
1.175				0.0008	0.0076	0.0224	0.0459	0.0787	0.1213	0.1738
1.200				0.0000	0.0034	0.0145	0.0340	0.0628	0.1011	0.1493
1.300						0.0000	0.0039	0.0162	0.0376	0.0685
1.400								0.0000	0.0043	0.0176
1.500										0.0000

Table of V(ϱ,η) (cont.)

η \ ϱ	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.000	1.7487	2.0441	2.3505	2.6632	2.9766	3.2840	3.5765	3.8419	4.0613	4.1888
0.025	1.7480	2.0432	2.3494	2.6618	2.9750	3.2819	3.5738	3.8382	4.0555	4.1758
0.050	1.7459	2.0406	2.3461	2.6578	2.9699	3.2756	3.5656	3.8271	4.0369	4.1523
0.075	1.7423	2.0362	2.3406	2.6510	2.9615	3.2650	3.5518	3.8078	4.0021	4.1221
0.100	1.7373	2.0299	2.3329	2.6414	2.9496	3.2499	3.5321	3.7788	3.9601	4.0867
0.125	1.7308	2.0218	2.3229	2.6290	2.9341	3.2302	3.5057	3.7367	3.9138	4.0470
0.150	1.7228	2.0119	2.3106	2.6137	2.9149	3.2054	3.4710	3.6878	3.8642	4.0034
0.175	1.7133	2.0001	2.2958	2.5953	2.8916	3.1749	3.4253	3.6349	3.8118	3.9565
0.200	1.7023	1.9862	2.2786	2.5737	2.8640	3.1373	3.3733	3.5789	3.7569	3.9066
0.225	1.6896	1.9704	2.2587	2.5486	2.8314	3.0899	3.3174	3.5206	3.6998	3.8540
0.250	1.6753	1.9523	2.2360	2.5196	2.7925	3.0369	3.2587	3.4602	3.6407	3.7990
0.275	1.6592	1.9321	2.2103	2.4864	2.7450	2.9803	3.1979	3.3980	3.5798	3.7417
0.300	1.6413	1.9094	2.1813	2.4476	2.6924	2.9210	3.1352	3.3344	3.5174	3.6825
0.325	1.6214	1.8840	2.1484	2.4010	2.6365	2.8598	3.0710	3.2694	3.4534	3.6214
0.350	1.5995	1.8558	2.1106	2.3499	2.5781	2.7969	3.0056	3.2032	3.3881	3.5587
0.375	1.5753	1.8241	2.0659	2.2957	2.5179	2.7327	2.9391	3.1359	3.3216	3.4944
0.400	1.5485	1.7881	2.0170	2.2393	2.4563	2.6675	2.8717	3.0677	3.2540	3.4287
0.425	1.5187	1.7459	1.9654	2.1813	2.3935	2.6014	2.8036	2.9987	3.1854	3.3617
0.450	1.4850	1.6999	1.9118	2.1220	2.3299	2.5346	2.7347	2.9290	3.1159	3.2937
0.475	1.4458	1.6515	1.8567	2.0617	2.2655	2.4672	2.6654	2.8587	3.0456	3.2245
0.500	1.4033	1.6013	1.8006	2.0007	2.2007	2.3994	2.5956	2.7878	2.9746	3.1545
0.525	1.3586	1.5499	1.7436	1.9391	2.1354	2.3313	2.5254	2.7165	2.9031	3.0836
0.550	1.3124	1.4976	1.6861	1.8772	2.0699	2.2629	2.4550	2.6449	2.8310	3.0120
0.575	1.2651	1.4446	1.6282	1.8151	2.0043	2.1945	2.3845	2.5730	2.7585	2.9398
0.600	1.2171	1.3912	1.5701	1.7529	1.9386	2.1259	2.3138	2.5008	2.6857	2.8670
0.625	1.1686	1.3375	1.5119	1.6907	1.8729	2.0575	2.2431	2.4286	2.6126	2.7937
0.650	1.1198	1.2838	1.4537	1.6286	1.8074	1.9891	2.1725	2.3563	2.5393	2.7201
0.675	1.0709	1.2301	1.3957	1.5667	1.7421	1.9209	2.1020	2.2840	2.4659	2.6462
0.700	1.0221	1.1766	1.3380	1.5051	1.6772	1.8530	2.0317	2.2119	2.3924	2.5721
0.725	0.9734	1.1235	1.2806	1.4439	1.6126	1.7855	1.9616	2.1399	2.3190	2.4979
0.750	0.9251	1.0707	1.2236	1.3832	1.5484	1.7183	1.8919	2.0681	2.2457	2.4236
0.775	0.8773	1.0184	1.1672	1.3230	1.4847	1.6516	1.8226	1.9966	2.1726	2.3493
0.800	0.8299	0.9667	1.1114	1.2634	1.4217	1.5854	1.7537	1.9254	2.0996	2.2751
0.825	0.7832	0.9157	1.0563	1.2044	1.3592	1.5198	1.6853	1.8547	2.0270	2.2011
0.850	0.7372	0.8654	1.0019	1.1462	1.2975	1.4549	1.6175	1.7845	1.9548	2.1273
0.875	0.6920	0.8159	0.9484	1.0889	1.2365	1.3907	1.5504	1.7148	1.8829	2.0538
0.900	0.6477	0.7673	0.8957	1.0323	1.1764	1.3272	1.4840	1.6457	1.8116	1.9806
0.925	0.6043	0.7197	0.8440	0.9767	1.1172	1.2646	1.4183	1.5773	1.7408	1.9079
0.950	0.5619	0.6730	0.7933	0.9221	1.0588	1.2028	1.3534	1.5096	1.6706	1.8356
0.975	0.5206	0.6275	0.7436	0.8685	1.0015	1.1420	1.2893	1.4426	1.6011	1.7639
1.000	0.4804	0.5831	0.6951	0.8160	0.9452	1.0822	1.2262	1.3766	1.5324	1.6929
1.025	0.4415	0.5399	0.6477	0.7647	0.8901	1.0234	1.1641	1.3113	1.4644	1.6225
1.050	0.4038	0.4979	0.6016	0.7145	0.8361	0.9658	1.1030	1.2471	1.3973	1.5528
1.075	0.3674	0.4573	0.5568	0.6656	0.7833	0.9093	1.0430	1.1838	1.3311	1.4840
1.100	0.3325	0.4181	0.5134	0.6181	0.7318	0.8540	0.9841	1.1216	1.2658	1.4160
1.125	0.2989	0.3802	0.4713	0.5719	0.6815	0.7999	0.9265	1.0606	1.2016	1.3489
1.150	0.2669	0.3439	0.4307	0.5271	0.6327	0.7472	0.8700	1.0006	1.1384	1.2828
1.175	0.2364	0.3090	0.3916	0.4837	0.5853	0.6958	0.8149	0.9419	1.0764	1.2177
1.200	0.2075	0.2758	0.3540	0.4419	0.5393	0.6459	0.7611	0.8845	1.0156	1.1537
1.300	0.1091	0.1597	0.2203	0.2909	0.3713	0.4612	0.5605	0.6686	0.7852	0.9097
1.400	0.0404	0.0730	0.1155	0.1681	0.2307	0.3033	0.3855	0.4772	0.5780	0.6876
1.500	0.0046	0.0187	0.0427	0.0767	0.1208	0.1751	0.2394	0.3136	0.3975	0.4907
1.600		0.0000	0.0049	0.0197	0.0446	0.0798	0.1253	0.1810	0.2468	0.3224
1.700				0.0000	0.0051	0.0205	0.0462	0.0825	0.1291	0.1861
1.800						0.0000	0.0053	0.0211	0.0476	0.0848
1.900								0.0000	0.0054	0.0217
2.000										0.0000

Table of V(ϱ,η) (cont.)

η \ ϱ	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50
0.050	4.1888									
0.075	4.1814									
0.100	4.1630	4.1888								
0.125	4.1370	4.1829								
0.150	4.1050	4.1675	4.1888							
0.175	4.0679	4.1444	4.1838							
0.200	4.0266	4.1151	4.1699	4.1888						
0.225	3.9816	4.0806	4.1489	4.1842						
0.250	3.9332	4.0416	4.1218	4.1716	4.1888					
0.275	3.8820	3.9985	4.0893	4.1521	4.1846					
0.300	3.8280	3.9519	4.0521	4.1265	4.1728	4.1888				
0.325	3.7716	3.9021	4.0108	4.0957	4.1544	4.1849				
0.350	3.7131	3.8495	3.9658	4.0601	4.1301	4.1737	4.1888			
0.375	3.6526	3.7942	3.9175	4.0203	4.1006	4.1562	4.1851			
0.400	3.5902	3.7366	3.8661	3.9767	4.0663	4.1329	4.1745	4.1888		
0.425	3.5262	3.6769	3.8120	3.9296	4.0278	4.1045	4.1576	4.1852		
0.450	3.4607	3.6152	3.7554	3.8794	3.9854	4.0713	4.1352	4.1750	4.1888	
0.475	3.3938	3.5517	3.6965	3.8264	3.9395	4.0339	4.1076	4.1588	4.1854	
0.500	3.3257	3.4866	3.6356	3.7708	3.8904	3.9926	4.0754	4.1371	4.1755	4.1888
0.525	3.2565	3.4201	3.5727	3.7127	3.8383	3.9477	4.0390	4.1103	4.1598	4.1855
0.550	3.1862	3.3522	3.5082	3.6525	3.7836	3.8995	3.9986	4.0789	4.1387	4.1759
0.575	3.1151	3.2831	3.4420	3.5904	3.7264	3.8484	3.9546	4.0433	4.1126	4.1606
0.600	3.0432	3.2129	3.3745	3.5264	3.6669	3.7945	3.9073	4.0037	4.0819	4.1400
0.625	2.9706	3.1417	3.3056	3.4607	3.6054	3.7380	3.8570	3.9605	4.0470	4.1145
0.650	2.8974	3.0697	3.2356	3.3935	3.5419	3.6792	3.8038	3.9140	4.0082	4.0845
0.675	2.8237	2.9970	3.1646	3.3250	3.4768	3.6183	3.7481	3.8644	3.9657	4.0502
0.700	2.7496	2.9236	3.0926	3.2552	3.4100	3.5555	3.6900	3.8120	3.9199	4.0120
0.725	2.6752	2.8496	3.0198	3.1844	3.3418	3.4908	3.6296	3.7569	3.8710	3.9702
0.750	2.6005	2.7752	2.9463	3.1125	3.2724	3.4245	3.5673	3.6994	3.8191	3.9250
0.775	2.5257	2.7004	2.8722	3.0398	3.2017	3.3566	3.5031	3.6396	3.7647	3.8767
0.800	2.4508	2.6254	2.7976	2.9663	3.1300	3.2875	3.4372	3.5778	3.7077	3.8255
0.825	2.3758	2.5501	2.7226	2.8921	3.0574	3.2171	3.3697	3.5140	3.6485	3.7716
0.850	2.3010	2.4747	2.6473	2.8174	2.9840	3.1456	3.3009	3.4485	3.5871	3.7151
0.875	2.2262	2.3993	2.5717	2.7423	2.9098	3.0731	3.2307	3.3814	3.5238	3.6564
0.900	2.1517	2.3239	2.4959	2.6667	2.8351	2.9998	3.1595	3.3129	3.4587	3.5955
0.925	2.0775	2.2486	2.4201	2.5909	2.7598	2.9257	3.0872	3.2430	3.3919	3.5326
0.950	2.0036	2.1735	2.3443	2.5149	2.6842	2.8509	3.0139	3.1720	3.3237	3.4678
0.975	1.9301	2.0987	2.2686	2.4388	2.6082	2.7756	2.9399	3.0998	3.2541	3.4014
1.000	1.8571	2.0242	2.1930	2.3627	2.5320	2.6999	2.8652	3.0267	3.1832	3.3335
1.025	1.7847	1.9501	2.1177	2.2866	2.4556	2.6238	2.7899	2.9528	3.1113	3.2641
1.050	1.7128	1.8764	2.0427	2.2107	2.3793	2.5475	2.7141	2.8782	3.0384	3.1935
1.075	1.6417	1.8033	1.9681	2.1349	2.3029	2.4709	2.6380	2.8029	2.9645	3.1217
1.100	1.5712	1.7309	1.8939	2.0595	2.2266	2.3943	2.5615	2.7271	2.8900	3.0490
1.125	1.5016	1.6590	1.8203	1.9844	2.1506	2.3177	2.4849	2.6509	2.8147	2.9753
1.150	1.4328	1.5879	1.7472	1.9098	2.0748	2.2412	2.4081	2.5744	2.7390	2.9008
1.175	1.3650	1.5177	1.6748	1.8357	1.9994	2.1649	2.3313	2.4976	2.6627	2.8256
1.200	1.2981	1.4482	1.6032	1.7622	1.9243	2.0888	2.2546	2.4207	2.5861	2.7498
1.300	1.0416	1.1803	1.3250	1.4752	1.6299	1.7884	1.9499	2.1135	2.2781	2.4430
1.400	0.8054	0.9309	1.0635	1.2027	1.3478	1.4980	1.6526	1.8108	1.9717	2.1346
1.500	0.5929	0.7037	0.8225	0.9489	1.0822	1.2219	1.3673	1.5176	1.6722	1.8301
1.600	0.4077	0.5022	0.6056	0.7175	0.8373	0.9644	1.0984	1.2386	1.3843	1.5347
1.700	0.2531	0.3300	0.4165	0.5122	0.6167	0.7295	0.8501	0.9780	1.1126	1.2532
1.800	0.1324	0.1904	0.2586	0.3367	0.4242	0.5210	0.6264	0.7401	0.8614	0.9900
1.900	0.0488	0.0868	0.1353	0.1943	0.2635	0.3425	0.4310	0.5287	0.6350	0.7494
2.000	0.0056	0.0222	0.0499	0.0885	0.1379	0.1977	0.2677	0.3477	0.4371	0.5355

Table of V(ϱ,η) (cont.)

η \ ϱ	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00
0.6	4.1763	4.1888								
0.7	4.0867	4.1422	4.1769	4.1888						
0.8	3.9296	4.0185	4.0905	4.1440	4.1773	4.1888				
0.9	3.7218	3.8363	3.9374	4.0236	4.0935	4.1453	4.1777	4.1888		
1.0	3.4761	3.6098	3.7333	3.8451	3.9437	4.0278	4.0959	4.1465	4.1779	4.1888
1.1	3.2029	3.3505	3.4906	3.6218	3.7428	3.8524	3.9490	4.0314	4.0980	4.1474
1.2	2.9107	3.0676	3.2194	3.3648	3.5027	3.6319	3.7509	3.8586	3.9535	4.0344
1.3	2.6070	2.7691	2.9283	3.0835	3.2334	3.3771	3.5132	3.6405	3.7578	3.8639
1.4	2.2983	2.4621	2.6249	2.7857	2.9435	3.0972	3.2456	3.3876	3.5222	3.6480
1.5	1.9906	2.1528	2.3158	2.4787	2.6405	2.8002	2.9567	3.1091	3.2562	3.3969
1.6	1.6892	1.8469	2.0070	2.1687	2.3311	2.4932	2.6541	2.8128	2.9683	3.1195
1.7	1.3991	1.5497	1.7041	1.8617	2.0215	2.1828	2.3446	2.5060	2.6662	2.8240
1.8	1.1250	1.2660	1.4122	1.5629	1.7174	1.8748	2.0344	2.1953	2.3566	2.5174
1.9	0.8715	1.0006	1.1361	1.2775	1.4239	1.5747	1.7292	1.8865	2.0459	2.2064
2.0	0.6426	0.7577	0.8804	1.0101	1.1460	1.2877	1.4344	1.5853	1.7398	1.8970

Table of V(ϱ,η) (cont.)

η \ ϱ	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
1.1	4.1782	4.1888								
1.2	4.0997	4.1482	4.1784	4.1888						
1.3	3.9574	4.0369	4.1012	4.1489	4.1786	4.1888				
1.4	3.7638	3.8685	3.9607	4.0392	4.1026	4.1495	4.1787	4.1888		
1.5	3.5300	3.6545	3.7691	3.8725	3.9636	4.0411	4.1037	4.1501	4.1789	4.1888
1.6	3.2655	3.4050	3.5370	3.6602	3.7737	3.8761	3.9662	4.0429	4.1047	4.1505
1.7	2.9786	3.1288	3.2737	3.4122	3.5431	3.6654	3.7778	3.8793	3.9685	4.0444
1.8	2.6769	2.8340	2.9877	3.1371	3.2811	3.4186	3.5486	3.6699	3.7815	3.8821
1.9	2.3673	2.5277	2.6865	2.8429	2.9960	3.1446	3.2877	3.4244	3.5536	3.6741
2.0	2.0562	2.2165	2.3770	2.5369	2.6952	2.8510	3.0034	3.1513	3.2937	3.4297

Table of V(ϱ,η) (cont.)

η \ ϱ	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	3.00
1.6	4.1790	4.1888								
1.7	4.1056	4.1510	4.1791	4.1888						
1.8	3.9706	4.0458	4.1064	4.1513	4.1792	4.1888				
1.9	3.7848	3.8847	3.9725	4.0470	4.1072	4.1517	4.1793	4.1888		
2.0	3.5580	3.6778	3.7878	3.8870	3.9741	4.0482	4.1078	4.1520	4.1794	4.1888

For the computation of the complete elliptic integrals of the first and second kinds, $K(k)$ and $E(k)$, we made use of approximations given by Hastings, [5].

Owing to the definition of Heuman's lambda function, the computation of this function could be reduced to the computation of incomplete elliptic integrals of the first and second kinds. These latter functions may suitably be computed by means of the strongly convergent expansions for these functions derived by Van Veen, [6].

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